Math 255A' Lecture 18 Notes

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1 Approximation and Eigenvalues of Compact Operators

1.1 Approximation of compact operators by finite rank operators

Last time, we were talking about invariant and reducing subspaces of a Hilbert space M. Here, we have $H = M \oplus M^\perp$ and $A \in \mathcal{B}(H)$ is

$$A = \begin{bmatrix} X & Y \\ W & Z \end{bmatrix}.$$

We saw that M is reducing $\iff P_M A = A P_M \iff Y = 0, W = 0$.

Proposition 1.1. *M is reducing if and only if M is invariant under both A and A^*.*

Proof. This is because

$$A^* = \begin{bmatrix} X^* & W^* \\ Y^* & Z^* \end{bmatrix}.$$

Then M is invariant for A^* iff $Y = 0$ iff $M^\perp$ is invariant for A. $\square$

Recall that $\mathcal{B}_0(H)$ is the space of compact operators, and $\mathcal{B}_{00}$ is the space of finite rank operators.

Theorem 1.1. *$\mathcal{B}_{00}(H)$ is dense in $\mathcal{B}_0(H)$.*

Proof. If $T \in \mathcal{B}_0(H)$, then $\overline{T(B_H)}$ is a compact metric space. So it is countable. Then $\operatorname{ran} T \subseteq \operatorname{span} \overline{T(B_H)} \subseteq \overline{\operatorname{span}} D$, where D is any countable dense set in $\overline{T(B_H)}$. So there is an orthonormal $\langle e_n \rangle_n$ such that $\operatorname{ran} T \subseteq \overline{\operatorname{span}}\{e_n\}$. Let P_m be the projection onto $\operatorname{span}\{e_1, \dots, e_m\}$. We will show that $\|P_m T - T\|_{\mathrm{op}} \to 0$.

Observe that for any $h \in \overline{B_H}$, we have $Th = \sum_n \langle Th, e_j \rangle e_h$. Then $P_n Th \to T_h$ in the norm of H. Let $\varepsilon > 0$. We can choose $h_1, \dots, j_k \in \overline{B_H}$ such that for all $h \in \overline{B_H}$, there is some i such that $\|Th - Th_i\| < \varepsilon$. Choose m such that $\|P_m Th_i - Th_i\| < \varepsilon$ for all $i = 1, \dots, k$. Then

$$\|P_m Th - Th\| < \|P_m(Th - Th_i)\| + \|P_m Th_i - Th_i\| + \|Th - i - Th\| < 3\varepsilon.$$

So $\|P_m T - T\|_{\mathrm{op}} < 3\varepsilon$. $\square$

Remark 1.1. If you try to do this with general Banach spaces, it fails. The issue is that you cannot guarantee that $\|P_m\| = 1$ for all m. So you lose control of the bound at the end.

Suppose $\langle e_n \rangle_n$ is an orthonormal basis for H. Define an operator by $Te_n = \alpha_n e_n$ for $\alpha_n \in \mathbb{F}$.

Lemma 1.1. *$T \in \mathcal{B}_0(H)$ if and only if $|\alpha_n| \to 0$.*

Proof. ($\implies$): Assume there exist some $\varepsilon > 0$ and $n_1 < n_2 < \cdots$ such that$|\alpha_{n_i} > \varepsilon$. Then $\{Te_{n_1}, Te_{n_2}, \cdots = \{\alpha_{n_1} e_n, \alpha_{n_2} e_{n_2}, \dots\} \subseteq T(\overline{B_H})$. These all are distance $\geq \varepsilon$ to each other and are orthonormal to each other.

($\impliedby$): Let

$$T_m e - n = \begin{cases} \alpha_n e_n & n \leq m \\ 0 & n > m \end{cases} = P_m T.$$

Then we have the diagonal matrix:

$$T - P_m T = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \alpha_{m+1} & & \\ & & & & \alpha_{m+2} & \\ & & & & & \ddots \end{bmatrix}.$$

So we can see that $\|T - P_m T\|_{\text{op}} \leq \max_{n>m} |\alpha_n|$. $\square$

Example 1.1. Let $k \in L^2(\mu \times \mu)$ and let

$$Kf(x) = \int k(x, y) f(y) \, d\mu(y).$$

For example, if $h \in L^2(-\pi, \pi)$, we have

$$Kf(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} h(x - y) f(y) \, dy.$$

Let the Fourier basis be $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{-inx}$ for all $n \in \mathbb{Z}$. Then we can check

$$Ke_n(x) = \widehat{h}(x) \cdot e_n(x).$$

1.2 Eigenvalues of compact operators

Definition 1.1. If $A \in \mathcal{B}(H)$, an **eigenvalue** of A is a $\lambda \in \mathbb{F}$ such that $\ker(A - \lambda) \neq \{0\}$. The **$\lambda$-eigenspace** is the set of **eigenvectors** corresponding to the eigenvalue λ. We denote the **point spectrum** $\sigma_p(A)$ to be the set of eigenvalues of A.

Remark 1.2. This is a special subset of the **spectrum**, which is the set of $\lambda \in \mathbb{F}$ such that $A - \lambda 1$ is not invertible.

Example 1.2. In $\mathbb{C}^4$, the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has no nonzero eigenvalues but is nonzero. This kind of phenomenon becomes much richer in infinite dimensions, and we can have compact operators with no nonzero eigenvalues but with interesting properties.

Example 1.3. On $L^2([0,1])$, the **Volterra operator** is

$$Vf(x) = \int_0^x f(y)\, dy = \int_0^1 \mathbb{1}_{\{y \leq x\}} f(y)\, dy.$$

Proposition 1.2. *The Volterra operator is compact but has no eigenvalues.*

Proof. Suppose $Vf = \lambda f$ with $f \neq 0$. If $\lambda = 0$, then the integral of f only any interval is 0, so $f = 0$. Suppose $\lambda \neq 0$. Then we get $f(x) = \lambda^{-1} \int_0^x f$, so f is absolutely continuous, f' exists, and $f'(x) = \lambda^{-1} f(x)$ a.e. Since we must have f is continuous, this gives $f'(x) = \lambda^{-1} f(x)$ everywhere. The solution to this differential equation is $f(x) = Ce^{cx}$. But we must have $C = 0$ because the original equation implies $f(0) = 0$. So $f = 0$. $\square$

Proposition 1.3. *Let $T \in \mathcal{B}_0(H)$ and $\lambda \in \sigma_p(T) \setminus \{0\}$. Then* $\dim \ker(T - \lambda 1) < \infty$.

Proof. Call $M = \ker(T - \lambda 1)$. Then $Tx = \lambda x$ for all $x \in M$. We have $T(\overline{B}_H) \supseteq T(\overline{B}_H \cap M) = \lambda \overline{B}_M$, which is not totally bounded unless $\dim M < \infty$. $\square$

Proposition 1.4. *Let $T \in \mathcal{B}_0(H)$, and let $\lambda \neq 0$. Assume that*

$$\inf\{\|(T - \lambda)h\| : \|h\| = 1\} = 0.$$

Then $\lambda \in \sigma_p(T)$.

Remark 1.3. This says that "approximate eigenvalues" are actually eigenvalues for compact operators.

Proof. Choose $h_1, h_2, \dots$ with $\|h_n\| = 1$ such that $Th_n - \lambda h_h = (T - \lambda)h_n \to 0$ in $\|\cdot\|$. Choose $n_1 < n_2 < \cdots$ susch that $Th_n \to g$. Then $\lambda h_n = Th_n - (Th_n - \lambda h_n) \to g$, so $h_n \to \lambda^{-1} g$. So $Th_n \to \lambda^{-1} Tg = g$. $\square$

Corollary 1.1. *Let $T \in \mathcal{B}_0(H)$, and suppose that $\lambda \notin \sigma_p(T) \cap \{0\}$ and $\overline{\lambda} \notin \sigma_p(T^*)$. Then $T - \lambda$ is invertible.*

Remark 1.4. In fact, we will see that $\overline{\lambda} \notin \sigma_p(T^*)$ is implied by $\lambda \notin \sigma_p(T) \cap \{0\}$.

Proof. We know that $\ker(T - \lambda) = \{0\}$. On the other hand,

$$(\operatorname{ran}(T - \lambda))^\perp = \ker(T^* - \overline{\lambda}) = \{0\}.$$

To finish, we will show that $\operatorname{ran}(T - \lambda)$ is closed. $(T - \lambda)h = 0$ has no nonzero solutions, so there is a $c > 0$ such that $\|(T - \lambda)h\| \geq c\|h\|$ for all h. So $(T - \lambda)$ is an open mapping, which forces $\operatorname{ran}(T - \lambda)$ to be closed. $\square$

1.3 The spectral theorem for self-adjoint operators

We will prove the following theorem.

Theorem 1.2 (Spectral theorem for self-adjoint operators)**.** *Suppose T is comapct and self adjoint. Then*

1. *$\sigma_p(T)$ is countable.*
2. *If $\sigma_p(T) \setminus \{0\} = \{\lambda_1, \lambda_2, \dots\}$ and P_n is the projection onto $\ker(T - \lambda_n)$, then*
   - *$P_n P_m = P_m P_n = 0$ for all $m \neq n$ (i.e. $\ker(T - \lambda_n) \perp \ker(T - \lambda_m)$).*
   - *$\lambda_n \in \mathbb{R}$ for all n.*
   - *$T = \sum_{n=1}^\infty \lambda_n P_n$ in $\|\cdot\|_{\text{op}}$.*

This is an infinite-dimensional diagonalization of T.